ON THE METASTABLE HOMOTOPY OF MOD 2 MOORE SPACES

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Abstract. In this article, we study the exponents of metastable homotopy of mod 2 Moore spaces. Our result gives that the double loop space of $4n$-dimensional mod 2 Moore spaces has a multiplicative exponent 4 below the range of 4 times the connectivity. As a consequence, the homotopy groups of $4n$-dimensional mod 2 Moore spaces have an exponent 4 below the range of 4 times the connectivity.

1. Introduction

The metastable homotopy has been studied by various people with fruitful results [1, 3, 4, 11, 12, 14, 15, 17, 20] since the early 1950s. The descriptions on the lower metastable homotopy groups of the Moore spaces given by M. G. Barratt [1] in 1954 leaded to computational results announced in [20]. In this article, we consider the exponents of the metastable homotopy groups of mod 2 Moore spaces.

Let $P^n(2) = \Sigma^{n-2}\mathbb{R}\mathrm{P}^2$ be the n-dimensional mod 2 Moore space with $n \geq 3$. It is well known that $P^n(2)$ has a suspension exponent of 4, that is, the degree 4 map $[4]\colon P^n(2) \to P^n(2)$ is null homotopic. By the classical result of M. G. Barratt [2], the metastable homotopy of $P^n(2)$ has an exponent dividing 8. This leads to a natural question whether the metastable homotopy of $P^n(2)$ has an exponent of 4. The answer to this question is negative for the cases $n \equiv 2, 3 \mod 4$ because there are $\mathbb{Z}/8$-summands occurring in the lower metastable homotopy groups according to [9, 18, 20]. The purpose of this article is to give an affirmed answer to the above question for the case $n \equiv 0 \mod 4$ with $n > 5$.

Our answer to the question is actually given by showing that the double loop space $\Omega^2 P^n(2)$ has a multiplicative exponent 4 below the range roughly 4 times the connectivity in the case $n \equiv 0 \mod 4$ with $n > 5$. More explicitly, our main result is as follows.

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Theorem 1.1. *Let* $n \equiv 0 \mod 4$ *with* $n > 4$*. Then the power map* $4\colon \Omega^2 P^n(2) \to \Omega^2 P^n(2)$ *restricted to the skeleton* $\mathrm{sk}_{4n-9}(\Omega^2 P^n(2))$ *is null homotopic.*

As a consequence, we get an answer on the exponents of the homotopy groups up to the range, where for the first case $n = 4$, the homotopy groups $\pi_k(P^4(2))$ are known up to the range [21].

Corollary 1.2. *Let* $n \equiv 0 \mod 4$*. Then* $4 \cdot \pi_k(P^n(2)) = 0$ *for* $k \leq 4n - 7$. □

We should point out that it is unknown whether the homotopy groups of mod 2 Moore spaces have a bounded exponent. It is known that there are infinitely many $\mathbb{Z}/8$-summands (in different dimensions) occurring in the homotopy groups of mod 2 Moore spaces [9]. Our result shows that the first $\mathbb{Z}/8$-torsion should occur in the range at least 4 times the connectivity in the case $n \equiv 0 \mod 4$.

The methodology for proving Theorem 1.1 is briefly described as follows. We begin to use the Cohen groups for displaying the explicit obstructions to the 4-th power map on the single loop space $\Omega P^n(2)$. By using the shuffle relations and the Hopf invariants on general configuration spaces, the 4-th power on the double loop space $\Omega^2 P^n(2)$ up to the range is decomposed as a composite involving the Whitehead product. After handling the reduced evaluation map, Theorem 1.1 is then proved in the special case when the Whitehead square ω_{n-1} is divisible by 2 by Theorem 3.6. With having more lemmas on the Whitehead product on $P^{4n}(2)$, Theorem 1.1 is finally proved by Theorem 5.3. Here, the key lemma (Lemma 4.3) is a special property for $4n$-dimensional mod 2 Moore spaces, which is hinted by Mark Mahowald's result [13] that $[\iota_{4n-1}, \eta_{4n-1}] = 0$.

The article is organized as follows. In section 2, we discuss the 4-th power map on the single loop spaces and and double loop spaces. The reduced evaluation map on mod 2 Moore spaces is studied in section 3, where Theorem 3.6 is the special case of Theorem 1.1 in the case when the Whitehead square is divisible by 2. We give some lemmas in section 4. Theorem 1.1 is proved in section 5, where Theorem 5.3 is Theorem 1.1.

2. The 4-th power map on looped suspensions

In this section, we display the obstructions to the 4-th power map on $\Omega\Sigma^2 X$ and $\Omega^2\Sigma^2 X$ below four times the connectivity for spaces $\Sigma^2 X$ having suspension exponent 4.

2.1. **The obstructions to the 4-th power map on $\Omega\Sigma^2X$.** We use the Cohen groups [8] for computing the obstructions to the 4-th power map. Recall that the Cohen group $K_n^{\mathbb{Z}/4}=K_n^{\mathbb{Z}/4}(x_1,\ldots,x_n)$ is combinatorially defined by the generators given by the letters $x_1,\ldots,x_n$ with following relations:

(1). the iterated commutators $[[x_{i_1},x_{i_2}],\ldots,x_{i_t}]=1$ if $p=q$ for some $1\le p<q\le n$, where the commutator $[a,b]=a^{-1}b^{-1}ab$, and
(2). $x_i^4=1$ for $1\le i\le n$.

Let $d_i\colon K_n^{\mathbb{Z}/4}\to K_{n-1}^{\mathbb{Z}/4}$ be the group homomorphism such that $d_i(x_j)=x_j$ for $j<i$, $d_i(x_i)=1$ and $d_i(x_j)=x_{j-1}$ for $j>i$. The Cohen group $H_n^{\mathbb{Z}/4}$ is defined as the equalizer of $d_i\colon K_n^{\mathbb{Z}/4}\to K_{n-1}^{|\mathbb{Z}/4}$ for $1\le i\le n$. Namely, $H_n^{\mathbb{Z}/4}$ is the subgroup of $K_n^{\mathbb{Z}/4}$ consisting of the words $w\in K_n^{\mathbb{Z}/4}$ with the property that $d_i(w)=d_1(w)$ for $1\le i\le n$. For the spaces Σ^2X satisfying the following hypothesis:

$$\text{the identity map } \mathrm{id}_{\Sigma^2X} \text{ is of order 4 in } [\Sigma^2X,\Sigma^2X], \tag{2.1}$$

there is a commutative diagram of groups

$$\begin{tikzcd}
K_n^{\mathbb{Z}/4} \arrow[r, "e_X"] & {[(\Sigma X)^{\times n},\Omega\Sigma^2X]} \\
H_n^{\mathbb{Z}/4} \arrow[u, hook] \arrow[r, "e_X"] & {[J_n(\Sigma X),\Omega\Sigma^2X],} \arrow[u, hook, "q_n^*"']
\end{tikzcd} \tag{2.2}$$

where $J(Y)$ is the James construction with the James filtration $J_n(Y)$, the monomorphism in the right column is induced by the quotient map $q_n\colon(\Sigma X)^{\times n}\to J_n(\Sigma X)$ and the group homomorphism e_X sends the letter x_i to the homotopy class of the composite

$$(\Sigma X)^{\times n}\xrightarrow{\pi_i}\Sigma X\hookrightarrow\Omega\Sigma^2X$$

with π_i the i-th coordinate projection for $1\le i\le n$. Let $\alpha_n=x_1x_2\cdots x_n\in H_n^{\mathbb{Z}/4}\le K_n^{\mathbb{Z}/4}$. Then $e_X(\alpha_n)$ is the homotopy class of the inclusion map $J_n(\Sigma X)\to\Omega\Sigma^2X$.

We are only interested in the range below four times the connectivity. It is sufficient to only consider α_3^4, which can be done by direct computations through the Magnus-type representation of $K_n^{\mathbb{Z}/4}$ into the non-commutative exterior algebra $A_n^{\mathbb{Z}/4}$. Here, $A_n^{\mathbb{Z}/4}=A_n^{\mathbb{Z}/4}(y_1,\ldots,y_n)$ is the quotient algebra of the tensor $T(y_1,\ldots,y_n)$ over $\mathbb{Z}/4$ subject to

the relations

$$y_{i_1}y_{i_2}\cdots y_{i_t}=0 \quad \text{if } p=q \text{ for some } 1\le p<q\le t. \tag{2.3}$$

The representation $e\colon K_n^{\mathbb{Z}/4}\to A_n^{\mathbb{Z}/4}$ is given by $e(x_i)=1+y_i$. It is proved in [8] that this is a faithful representation of $K_n^{\mathbb{Z}/4}$.

Now $e(\alpha_3^4)=((1+y_1)(1+y_2)(1+y_3))^4$ in $A_3^{\mathbb{Z}/4}$. Note that

$$(1+y_1)(1+y_2)(1+y_3)=1+\sigma_1+\sigma_2+\sigma_3,$$

where $\sigma_1=y_1+y_2+y_3$, $\sigma_2=y_1y_2+y_1y_3+y_2y_3$ and $\sigma_3=y_1y_2y_3$. Let $\Delta=\sigma_1+\sigma_2+\sigma_3$. Then

$$e(\alpha_3^4)=(1+\Delta)^4=1+4\Delta+6\Delta^2+4\Delta^3+\Delta^4=1+2\Delta^2$$

in $A_3^{\mathbb{Z}/4}$ because $4\alpha=0$ for $\alpha\in A_n^{\mathbb{Z}/4}$ and $\Delta^4\in I^4A_3^{\mathbb{Z}/4}=0$, the 4-fold product of the augmentation ideal $IA_3^{\mathbb{Z}/4}$. By using the property that $I^4A_3^{\mathbb{Z}/4}=0$, we have $\Delta^2=\sigma_1^2+\sigma_2\sigma_1+\sigma_1\sigma_2$. With taking the notation $[\alpha,\beta]=\alpha\beta-\beta\alpha$ for α,β in an algebra A and using the relations (2.3), we have

$$\begin{aligned}
2\sigma_1^2 &= 2(y_1+y_2+y_3)^2\\
&= 2(y_2y_1+y_3y_1+y_1y_2+y_3y_2+y_1y_3+y_2y_3)\\
&= 2([y_1,y_2]+[y_1,y_3]+[y_2,y_3]),
\end{aligned}$$

$$\begin{aligned}
2(\sigma_1\sigma_2+\sigma_2\sigma_1) &= 2(2y_1y_2y_3+y_2y_3y_1+y_1y_3y_2+y_2y_1y_3+y_3y_1y_2)\\
&= 2(y_2y_3y_1+y_1y_3y_2+y_2y_1y_3+y_3y_1y_2)\\
&= 2(y_2([y_1,y_3])+[y_1,y_3]y_2)\\
&= 2[[y_1,y_3],y_2].
\end{aligned}$$

(**Note.** Since we are working by modulo 4, the sign $\pm$ on the terms can be ignored after multiplying by 2.) Now, by using the property [22, Lemma 1.4.8] that $e([[x_{i_1},x_{i_2}],\ldots,x_{i_t}])=1+[[y_{i_1},y_{i_2}],\ldots,y_{i_t}]$, we have

$$\alpha_3^4=([x_1,x_2]^2[x_1,x_3]^2[x_2,x_3]^2)\cdot[[x_1,x_3],x_2]^2. \tag{2.4}$$

We give a remark that the above method is valid for computing α_n^4 for small n, more effective method for determining α_n^4 for general n can be seen in [16]. The geometric interpretation of formula (2.4) through the representation e_X gives the following lemma.

Lemma 2.1 (Obstruction Lemma)**.** *Let X be a CW-complex such that $4\cdot[\mathrm{id}_{\Sigma^2X}]=0$ in $[\Sigma^2X,\Sigma^2X]$. Let $4|_{J_3}\colon J_3(\Sigma X)\to\Omega\Sigma^2X$ be the restriction of the power map $4\colon J(\Sigma X)\simeq\Omega\Sigma^2X\to\Omega\Sigma^2X$. Then there is a decomposition*

$$[4|_{J_3}]=\zeta_2\cdot\zeta_3$$

in the group $[J_3(\Sigma X),\Omega\Sigma^2X]$, where ζ_2 is represented by the composite

$$J_3(\Sigma X)\hookrightarrow J(\Sigma X)\xrightarrow{H_2}J((\Sigma X)^{\wedge 2})\xrightarrow{\Omega W_2^2}\Omega\Sigma^2X$$

with H_k the k-th James-Hopf invariant and W_k the k-fold Whitehead product, and ζ_3 is represented by the composite

$$J_3(\Sigma X) \xrightarrow{\text{pinch}} (\Sigma X)^{\wedge 3} \xrightarrow{\tau_{2,3}} (\Sigma X)^{\wedge 3} \xrightarrow{W_3^2} \Omega\Sigma^2 X.$$

with $\tau_{2,3}$ the map switching positions 2 and 3 in the self-smash product. □

2.2. **The elimination of the obstruction ζ_3.** Consider the looping homomorphism

$$\Omega\colon [J_3(\Sigma X), \Omega\Sigma^2 X] \longrightarrow [\Omega J_3(\Sigma X), \Omega^2\Sigma^2 X], \quad [f] \mapsto [\Omega f].$$

The obstruction ζ_3 can be always eliminated after looping using the shuffle relations introduced in [22]. Here we give a proof by highlighting the ideas of the shuffle relations.

Proposition 2.2. *The element ζ_3 lies in the kernel of the looping homomorphism defined as above. Thus, for any space Z and any map $f\colon \Sigma Z \to J_3(\Sigma X)$, the composite $\zeta_3 \circ f\colon \Sigma Z \to \Omega\Sigma^2 X$ is null homotopic.*

Proof. Let $Y = \Sigma X$. Let $J(Y) \wedge J(Y)$ be filtered by

$$\mathrm{Fil}_n(J(Y) \wedge J(Y)) = \bigcup_{i+j \le n} J_i(Y) \wedge J_j(Y).$$

Since Y is a co-H-space, there exists a filtration-preserving map

$$\bar{\psi}\colon J(Y) \to J(Y) \wedge J(Y)$$

such that $\bar{\psi}$ is homotopic to the reduced diagonal $\bar{\Delta}$. Consider the following homotopy commutative diagram

$$\begin{tikzcd}
J(Y) \arrow[d, "\bar{\Delta}"] & J_3(Y) \arrow[l, hook'] \arrow[r, "\text{pinch}"] \arrow[d, "\bar{\psi}_3"'] & Y^{\wedge 3} \arrow[d, "\text{shuffle}"'] \\
J(Y)\wedge J(Y) \arrow[dr, dashed, "H"] & \mathrm{Fil}_3(J(Y)\wedge J(Y)) \arrow[l, hook'] \arrow[r, "\text{pinch}"] & (J_2(Y)/J_1(Y)\wedge Y)\vee(Y\wedge J_2(Y)/J_1(Y)) \arrow[d, "\text{proj.}"] \\
 & J(Y^{\wedge 3}) \arrow[dr, "\Omega W_3"] & J_2(Y)/J_1(Y)\wedge Y = Y^{\wedge 3} \arrow[l, hook'] \arrow[d, "S_3"] \\
 & & \Omega\Sigma Y,
\end{tikzcd}$$

where S_3 is the Samelson product and the extension map H exists by the suspension splitting of $J(Y)\wedge J(Y)$. By using the Cohen program, the composite from $J_3(Y)$ goes through the right column represents the element

$$[[x_1,x_2],x_3]+[[x_2,x_1],x_3]+[[x_3,x_1],x_2]=-[[x_1,x_3],x_2]$$

in the Cohen group K_3. The assertion follows by letting the composite from $J_3(Y)$ take the path from the right hand side with using the property that $\Omega\bar{\Delta}\colon \Omega J(Y)\to\Omega(J(Y)\wedge J(Y))$ is null homotopic. $\square$

2.3. **The configuration spaces and the obstruction ζ_2.** The obstruction ζ_2 is essential after looping in general. We use configuration spaces for reducing the obstruction ζ_2 from the cubic range to the quadratic range. We refer C. -F. Bödigheimer's work [5] as a reference on configuration space models for mapping spaces as well as his constructions of the Hopf invariants on configuration spaces.

Let M be a smooth manifold, M_0 a submanifold, and X a pointed CW-complex. Let $C(M,M_0;X)$ be the configuration spaces with labels in X in the sense of [5] with the filtration $C_n=C_n(M,M_0;X)$ induced by the configuration length. Let $D_n=D_n(M,M_0;X)$ denote C_n/C_{n-1}. We will use the following properties:

(1). [5, Lemma, p. 178] Let N be a codimension zero submanifold M. Then the isotopy cofibration

$$(N, N\cap M_0) \longrightarrow (M, M_0) \longrightarrow (M, N\cup M_0)$$

induces a quasi-fibration

$$C(N, N\cap M_0; X) \longrightarrow C(M, M_0; X) \longrightarrow C(M, N\cup M_0; X)$$

provided that $(N, N\cap N_0)$ or X is connected.

(2). [5, Section 3] Let $V = \bigvee_{k=1}^{\infty} D_k$. There is a Cohen construction [6] as a power set map $P\colon C(M, M_0; X) \to C(\mathbb{R}^\infty; V)$, which is natural on (M, M_0) and X, inducing a stable splitting of $C(M, M_0; X)$. The Hopf invariant is given by the composite

$$H_k\colon C(M, M_0; X) \xrightarrow{P} C(\mathbb{R}^\infty; V) \xrightarrow{\text{proj.}} C(\mathbb{R}^\infty; D_k)$$

for $k \geq 1$.

In particular, let $I = [0,1]$, there is a quasi-fibration

$$C([0,1]\times I; X) \longrightarrow C(([0,3],[2,3])\times I; X) \longrightarrow C(([0,3],[0,1]\cup[2,3])\times I; X)$$

for any path-connected CW-complex X with

$$C(([0,3],[2,3])\times I; X) \simeq * \text{ and } C([0,1]\times I; X) \simeq \Omega^2\Sigma^2 X.$$

We choose the evaluation map

$$\Sigma\Omega^2\Sigma^2 X \simeq \Sigma C(I^2; X) \to \Omega\Sigma^2 X \simeq J(\Sigma X)$$

as the composite of

$$\Sigma C(I^2; X) \simeq C(([0,3],\partial_+)\times I; X)/C(I\times I; X) \xrightarrow{\text{pinch}} C(([0,3],\partial)\times I; X),$$

where $\partial_+ = [2,3]$ and $\partial = [0,1]\cup[2,3]$, followed by composing the homotopy inverse of the composite

$$J(\Sigma X) \xhookrightarrow{\simeq} C(I; \Sigma X) \xhookrightarrow{\simeq} C(([0,3],\partial)\times I; X).$$

The evaluation map $\sigma\colon \Sigma C(I^2; X) \to J(\Sigma X)$ defined in such a way is a filtration-preserving map up to homotopy, and so its restrictions give maps

$$\sigma_k\colon \Sigma C_k(I^2; X) \longrightarrow J_k(\Sigma X) \tag{2.5}$$

inducing the *reduced evaluation maps*

(2.6)

$$\bar\sigma_k\colon \Sigma D_k(I^2; X) = \Sigma(C_k/C_{k-1}) \longrightarrow (\Sigma X)^{\wedge k} = J_k(\Sigma X)/J_{k-1}(\Sigma X)$$

for $k \geq 1$, where $\sigma_1 = \bar{\sigma}_1 \colon \Sigma C_1(I^2; X) = \Sigma X \to J_1(\Sigma X)$ is a homotopy equivalence. Moreover, by applying the naturality of the Hopf invariants on (M, M_0) to the composites in the definition of the evaluation map σ, there is a homotopy commutative diagram

(2.7)
$$\begin{tikzcd}
 & \Sigma C(I^2;X) \arrow[rr, "\Sigma H_k"] \arrow[dd, "\sigma"] & & \Sigma C(\mathbb{R}^\infty; D_k) \arrow[r, equal] & \Sigma C(\mathbb{R}^\infty; D_k) \arrow[dd, dashed] \\
\Sigma C_k \arrow[ur, hook] \arrow[rr, "\text{pinch}"] \arrow[dd, "\sigma_k"] & & \Sigma D_k \arrow[ur, hook] \arrow[dd, "\bar{\sigma}_k"] & & \\
 & J(\Sigma X) \arrow[rr, "H_k"] & & J((\Sigma X)^{\wedge k}) \arrow[r, hook] & C(\mathbb{R}^\infty; (\Sigma X)^{\wedge k}) \\
J_k(\Sigma X) \arrow[ur, hook] \arrow[rr, "\text{pinch}"] & & (\Sigma X)^{\wedge k} \arrow[ur, hook] & &
\end{tikzcd}$$

Let $\mathrm{sk}_n(Y)$ denote the n-th skeleton of Y.

Theorem 2.3. *Let X be a simply connected space with the connectivity $|X|$ such that $\mathrm{id}_{\Sigma^2 X}$ has exponent 4 in $[\Sigma^2 X, \Sigma^2 X]$. Then there is a map*

$$\tilde{H}_2 \colon \mathrm{sk}_{4|X|-1}(\Omega^2 \Sigma^2 X) \to D_2(I^2; X)$$

such that the adjoint map of the 4-th power $4 \colon \Omega^2\Sigma^2 X \to \Omega^2 \Sigma^2 X$ restricted to $\mathrm{sk}_{4|X|-1}(\Omega^2\Sigma^2 X)$ is homotopic to the composite

$$\Sigma \mathrm{sk}_{4|X|-1}(\Omega^2\Sigma^2 X) \xrightarrow{\Sigma \tilde{H}_2} \Sigma D_2(I^2; X) \xrightarrow{\bar{\sigma}_2} (\Sigma X)^{\wedge 2} \xrightarrow{2 \cdot S_2} \Omega \Sigma^2 X.$$

Proof. Note that $D_2(I^2; X)$ is the $(4|X| - 1)$-skeleton of $C(\mathbb{R}^\infty; D_2)$. There is a homotopy commutative diagram

(2.8)
$$\begin{tikzcd}
\mathrm{sk}_{4|X|-1}(\Omega^2\Sigma^2 X) \arrow[r, hook] \arrow[d, dashed, "\tilde{H}_2"] & C_3(I^2; X) \arrow[r, hook] & \Omega^2\Sigma^2 X \arrow[d, "H_2"] \\
D_2 \arrow[rr, hook] & & C(\mathbb{R}^\infty; D_2)
\end{tikzcd}$$

for some map $\tilde{H}_2$. This gives the map $\tilde{H}_2$ in the statement.

By Proposition 2.2, $\zeta_3 \circ \sigma_3$ is null homotopic. So we only need to consider the obstruction ζ_2. By diagram (2.7) using the property that $|(\Sigma X)^{\wedge 2}| = 2(|X|+1)$, there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma(\mathrm{sk}_{4|X|-1}(\Omega^2\Sigma^2 X)) \arrow[r, "\sigma_3|"] \arrow[d, "\Sigma\tilde{H}_2"] & J_3(\Sigma X) \arrow[dr, "H_2"] & \\
\Sigma D_2(I^2; X) \arrow[r, "\bar{\sigma}_2"] & (\Sigma X)^{\wedge 2} \arrow[r, hook] & J((\Sigma X)^{\wedge 2}).
\end{tikzcd}$$

The assertion then follows by Lemma 2.1.

□

3. The reduced evaluation map on mod 2 Moore spaces

In this section, we give some lemmas on the reduced evaluation map

$$\bar{\sigma} = \bar{\sigma}_2\colon \Sigma D_2 = \Sigma D_2(I^2; X) = \Sigma D_2(\mathbb{R}^2; X) \longrightarrow (\Sigma X)^{\wedge 2}$$

in the case that X is a mod 2 Moore space. Let $P^n(2) = S^{n-1} \cup_2 e^n$ be the n-dimensional mod 2 Moore space.

Lemma 3.1. [21, Proposition 2.5] *Let $n \geq 3$. Then the degree 2 map $[2]\colon P^n(2) \to P^n(2)$ is homotopic to the composite*

$$P^n(2) \xrightarrow{\text{pinch}} S^n \xrightarrow{\eta} S^{n-1} \hookrightarrow P^n(2).$$

Thus the degree 2 map $[2]\colon P^n(2)^{\wedge 2} \to P^n(2)^{\wedge 2}$ is homotopic to the composite

$$P^n(2)^{\wedge 2} \xrightarrow{\text{pinch}} P^{2n}(2) \xrightarrow{\eta\wedge\mathrm{id}} P^{2n-1}(2) \hookrightarrow P^n(2)^{\wedge 2}.$$

□

Let u, v be a basis for the mod 2 homology $\tilde{H}_*(P^n(2))$ with $|u| = n-1$ and $|v| = n$. Then the mod 2 homology $H_*(\Omega P^{n+1}(2)) = T(u, v)$ with $Sq^1_* v = u$. By the work of Dyer-Lashof [10], $\tilde{H}_*(D_2(\mathbb{R}^2; P^{n-1}(2)))$ has the following basis

$$\begin{tikzcd}
2n-1 & & & \tau(v^2) \arrow[dd, "Sq^2_*"] \arrow[dr, "Sq^1_*"] & \\
2n-2 & & \tau(v)^2 \arrow[dd, "Sq^2_*"] \arrow[dr, "\beta_2"] & & \tau([u,v]) \\
2n-3 & \tau(u)\tau(v) \arrow[dr, "Sq^1_*"] & & \tau(u^2) & \\
2n-4 & & \tau(u)^2, & &
\end{tikzcd}$$

where the Steenrod operations follow from that on $H_*(\Omega P^{n+1}(2))$.

The reduced evaluation $\bar{\sigma}_*\colon \tilde{H}_*(D_2(\mathbb{R}^2, P^{n-1}(2))) \to \tilde{H}_*((P^n(2))^{\wedge 2})$ is given by $\bar{\sigma}_*(\tau(v^2)) = v^2$, $\bar{\sigma}_*(\tau[u,v]) = [u,v]$, $\bar{\sigma}_*(\tau(u^2)) = u^2$ and sending remaining 3 elements to zero.

Let us do cellular analysis on the homotopy of $\Sigma D_2(\mathbb{R}^2; P^{n-1}(2))$. The elements $\{\tau(v^2), \tau([u,v]), \tau(v)^2, \tau(u^2)\}$ has a structure of $P^{2n-2}(4)$ attached by 2-cells through a map $P^{2n-2}(2) \to P^{2n-2}(4)$.

Lemma 3.2. *There exists a unique 4-cell complex C^{2n-1} such that mod 2 homology $\tilde{H}_*(C^{2n-1})$ has a basis $\{a_{2n-3}, b_{2n-2}, c_{2n-2}, d_{2n-1}\}$ with $\beta_2(b) = a, Sq^2_*(d) = a$ and $Sq^1_*(d) = c$.*

Proof. Consider the short exact sequence

$$2 \cdot \pi_{2n-3}(P^{2n-2}(4)) = \mathbb{Z}/2 \twoheadleftarrow [P^{2n-2}(2), P^{2n-2}(4)] \hookleftarrow \pi_{2n-2}(P^{2n-2}(4)) = \mathbb{Z}/2.$$

Let g_1 be the map in the commutative diagram of cofibre sequences

$$\begin{tikzcd}
S^{2n-3} \arrow[r, hook] \arrow[d, "{[2]}"] & P^{2n-2}(2) \arrow[r] \arrow[d, "g_1"] & S^{2n-2} \arrow[d, equal] \\
S^{2n-3} \arrow[r, hook] & P^{2n-2}(4) \arrow[r] & S^{2n-2}.
\end{tikzcd}$$

Then $2[g]$ is given by the composite

$$P^{2n-2}(2) \longrightarrow S^{2n-2} \xrightarrow{\eta} S^{2n-3} \longrightarrow P^{2n-2}(2) \xrightarrow{g_1} P^{2n-2}(4),$$

which is null homotopic because $g_1|S^{2n-3}$ factors through degree $[2]\colon S^{2n-3}\to S^{2n-3}$. Then

$$[P^{2n-2}(2),P^{2n-2}(4)]=\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

Let g_2 be the composite

$$P^{2n-2}(2)\xrightarrow{q}S^{2n-2}\xrightarrow{\eta}S^{2n-3}\hookrightarrow P^{2n-2}(4).$$

Then the 3 essential elements in $[P^{2n-2}(2),P^{2n-2}(4)]$ are given by $[g_1]$, $[g_2]$ and $[g_1+g_2]$.

Since $[g_1]_*,[g_1+g_2]_*\colon H_{2n-2}(P^{2n-2}(2))\to H_{2n-2}(P^{2n-2}(4))$ are nonzero, $[g_2]$ is the only homotopy class as the attaching map for C^{2n-1}. The proof is finished.

□

By pinching two bottom elements, we have a pinch map $\phi\colon \Sigma D_2(\mathbb{R}^2;P^{n-1}(2))\to C^{2n}$ with a commutative diagram

$$\begin{tikzcd}
\Sigma D_2(\mathbb{R}^2;P^{n-1}(2)) \arrow[r, "\bar\sigma"] \arrow[d, "\phi"] & P^n(2)^{\wedge 2} \arrow[d, "p_2"] \\
C^{2n} \arrow[r, "p_3"] & P^{2n}(2),
\end{tikzcd}\tag{3.1}$$

where p_3 induces an epimorphism on mod 2 homology.

Consider the commutative diagram of cofibre sequences

$$\begin{tikzcd}
P^{2n-2}(2) \arrow[r, "g_2"] \arrow[d, equal] & P^{2n-2}(4) \arrow[r] & C^{2n-1} \arrow[r] & P^{2n-1}(2) \arrow[d, equal] \\
P^{2n-2}(2) \arrow[r, "\eta\circ q"] \arrow[d, equal] & S^{2n-3} \arrow[r] \arrow[u, hook] & \tilde C^{2n-1} \arrow[r] \arrow[u, "j_1"] & P^{2n-2}(2) \arrow[d, equal] \\
P^{2n-2}(2) \arrow[r, "\bar\eta\circ q"] & P^{2n-3}(2) \arrow[r] \arrow[u] & \bar C^{2n-1} \arrow[r] \arrow[u, "j_2"] & P^{2n-2}(2),
\end{tikzcd}$$

where $\bar\eta\colon S^{2n-2}\to P^{2n-3}(2)$ is a lifting of η. Observe that

(1). The map j_1 induces a monomorphism on mod 2 homology.

(2). The homotopy cofibre of $j_1\circ j_2\colon \bar C^{2n-1}\to C^{2n-1}$ is the same as the homotopy cofibre of $P^{2n-3}(2)\to S^{2n-3}\to P^{2n-2}(4)$.

(3). The mod 2 homology $\tilde{H}_*(\bar{C}^{2n-1})$ has a basis $\{x_{2n-1}, x_{2n-2}, x_{2n-3}, x_{2n-4}\}$ with $Sq^1_*(x_{2n-1}) = x_{2n-2}, Sq^1_*(x_{2n-3}) = x_{2n-4}, Sq^2_*(x_{2n-1}) = x_{2n-3}, Sq^2_*(x_{2n-2}) = x_{2n-4}$.

Lemma 3.3. *The homotopy cofibre of $P^{2n-3}(2) \to S^{2n-3} \to P^{2n-2}(4)$ is $P^{2n-2}(8)$.*

Proof. This follows from the commutative diagram of cofibre sequences

$$\begin{tikzcd}
 & P^{2n-3} \arrow[r, equal] \arrow[d] & P^{2n-3}(2) \arrow[d] \\
S^{2n-3} \arrow[r, "{[4]}"] \arrow[d, equal] & S^{2n-3} \arrow[r, hook] \arrow[d, "{[2]}"] & P^{2n-2}(4) \arrow[d] \\
S^{2n-3} \arrow[r, "{[8]}"] & S^{2n-3} \arrow[r] & P^{2n-2}(8).
\end{tikzcd}$$

□

Lemma 3.4. *The space $\bar{C}^{2n-1} \simeq \Sigma^{2n-7}\mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2$.*

Proof. We only need to show that there is a unique 4-cell complex having the same homology as $\bar{C}^{2n+1}$ with the same Steenrod module structure.

Consider the homotopy classed $[P^{2n-2}(2), P^{2n-3}(2)]$. There is a short exact sequence

$$\pi_{2n-3}(P^{2n-3}(2)) = \mathbb{Z}/2 \twoheadleftarrow [P^{2n-2}(2), P^{2n-3}(2)] \hookleftarrow \pi_{2n-2}(P^{2n-3}(2))/2.$$

There are 3 essential homotopy classes in $[P^{2n-2}(2), P^{2n-3}(2)]$ given as follows

(1). $\eta \wedge \mathrm{id}\colon P^{2n-2}(2) \to P^{2n-3}(2)$. The homology of its cofibre has the same structure as $\tilde{H}_*(\Sigma^{2n-7}\mathbb{C}\mathrm{P}^2 \wedge \mathbb{R}\mathrm{P}^2)$.

(2). The composite $h_1\colon P^{2n-2}(2) \xrightarrow{\text{pinch}} S^{2n-2} \xrightarrow{\bar{\eta}} P^{2n-3}(2)$. The reduced mod 2 homology of the cofibre C_{h_1} has a basis

$$\{y_{2n-1}, y_{2n-2}, y_{2n-3}, y_{2n-4}\}$$

with $Sq^1_*(y_{2n-1}) = y_{2n-2}, Sq^1_*(y_{2n-3}) = y_{2n-4}, Sq^2_*(y_{2n-1}) = y_{2n-3}$ and $Sq^2_*(y_{2n-2}) = 0$. Here $Sq^2_*(y_{2n-2}) = 0$ because $h_1|_{S^{2n-3}}$ is null homotopic.

(3). The composite $h_2\colon P^{2n-2}(2) \xrightarrow{\tilde{\eta}} S^{2n-4} \hookrightarrow P^{2n-3}(2)$, where $\tilde{\eta}$ is an extension of $\eta\colon S^{2n-3} \to S^{2n-4}$. The reduced mod 2 homology of the cofibre C_{h_2} has a basis

$$\{z_{2n-1}, z_{2n-2}, z_{2n-3}, z_{2n-4}\}$$

with $Sq^1_*(z_{2n-1}) = z_{2n-2}, Sq^1_*(z_{2n-3}) = z_{2n-4}, Sq^2_*(y_{2n-2}) = y_{2n-4}$ and $Sq^2_*(y_{2n-1}) = 0$. Here $Sq^2_*(y_{2n-2}) = 0$ because

$$P^{2n-2}(2) \xrightarrow{h_2} P^{2n-3}(2) \to S^{2n-3}$$

is null homotopic.

The proof is finished.

□

The following lemma will be useful.

Lemma 3.5. *There is a commutative diagram*

$$\begin{tikzcd} C^{2n} \arrow[r, "p_3"] \arrow[d] & P^{2n}(2) \arrow[d, "\eta\wedge\mathrm{id}"] \\ P^{2n-1}(8) \arrow[r, "\rho"] & P^{2n-1}(2), \end{tikzcd}$$

where $\rho_*\colon H_{2n-2}(P^{2n-1}(8);\mathbb{Z}/2) \to H_{2n-2}(P^{2n-2}(2);\mathbb{Z}/2)$ is an isomorphism.

Proof. Consider the commutative diagram of cofibre sequences

$$\begin{tikzcd} \bar{C}^{2n} \arrow[r, "j_1\circ j_2"] \arrow[d, equal] & C^{2n} \arrow[r] \arrow[d, "\mathrm{pinch}"] & P^{2n-1}(8) \arrow[d, "\rho"] \\ \bar{C}^{2n} \arrow[r] & P^{2n}(2) \arrow[r, "h"] & P^{2n-1}(2). \end{tikzcd}$$

By Lemma 3.4, $h \simeq \eta\wedge\mathrm{id}$ and hence the result. □

Let us first prove the special case when the Whitehead square is divisible by 2.

Theorem 3.6. *Let $n+1 \equiv 0 \mod 4$ with $n+1 > 5$. Suppose that the Whitehead square ω_n is divisible by 2. Then the 4-power map $4|\colon \Omega^2 P^{n+1}(2) \to \Omega^2 P^{n+1}(2)$ restricted to the skeleton $\mathrm{sk}_{4(n-1)-1}$ is null homotopic.*

Proof. By Theorem 2.3, it suffices to show that the composite

$$\Sigma D_2 \xrightarrow{\bar\sigma_2} (P^n(2))^{\wedge 2} \xrightarrow{[2]} (P^n(2))^{\wedge 2} \xrightarrow{S_2} \Omega P^{n+1}(2)$$

is null homotopic. By Lemma 3.1, it suffices to prove that the composite

$$\Sigma D_2 \xrightarrow{\bar\sigma_2} (P^n(2)^{\wedge 2} \xrightarrow{\text{pinch}} P^{2n}(2) \xrightarrow{\eta\wedge\mathrm{id}} P^{2n-1}(2) \xrightarrow{S_2|} \Omega P^{n+1}(2) \tag{3.2}$$

is null homotopic.

Since ω_n is divisible by 2,

$$S_2|_{S^{2n-2}}\colon S^{2n-2} \to \Omega P^{n+1}(2)$$

is null homotopic. Thus $S_2|\colon P^{2n-1}(2) \to \Omega P^{n+1}(2)$ factors through S^{2n-1}.

Let $\rho\colon P^{2n-1}(8) \to P^{2n-1}(2)$ be the map in Lemma 3.5. We show that there is a commutative diagram of cofibre sequences

$$\begin{tikzcd}
S^{2n-2} \arrow[r, hook] \arrow[d, equal] & P^{2n-1}(8) \arrow[r] \arrow[d, "\rho"] & S^{2n-1} \arrow[d, "{[4]}"] \\
S^{2n-2} \arrow[r] & P^{2n-1}(2) \arrow[r] & S^{2n-1}.
\end{tikzcd} \tag{3.3}$$

There is a short exact sequence

$$\pi_{2n-2}(P^{2n-1}(2)) = \mathbb{Z}/2 \twoheadleftarrow [P^{2n-1}(8), P^{2n-1}(2)] \hookleftarrow \pi_{2n-1}(P^{2n-1}(2)) = \mathbb{Z}/2$$

Let $\alpha\colon P^{2n-1}(8) \to P^{2n-1}(2)$ be an extension of the inclusion map $S^{2n-2} \to P^{2n-1}(2)$. Then $2[\alpha] = 0$. Thus

$$[P^{2n-1}(8), P^{2n-1}(2)] = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Let β be the composite

$$P^{2n-1}(8) \xrightarrow{\text{pinch}} S^{2n-1} \xrightarrow{\eta} S^{2n-2} \hookrightarrow P^{2n-2}(2).$$

Then $[P^{2n-1}(8), P^{2n-1}(2)]$ is generated by α and β. We can make a choice of α from the commutative diagram

$$\begin{tikzcd}
S^{2n-2} \arrow[r, "{[8]}"] \arrow[d, "{[4]}"] & S^{2n-2} \arrow[r] \arrow[d, equal] & P^{2n-1}(8) \arrow[r] \arrow[d, "\alpha"] & S^{2n-1} \arrow[d, "{[4]}"] \\
S^{2n-2} \arrow[r, "{[2]}"] & S^{2n-2} \arrow[r] & P^{2n-1}(2) \arrow[r, "p_4"] & S^{2n-1}.
\end{tikzcd}$$

Then $[\rho] = [\alpha]$ or $[\alpha + \beta]$. Since $p_{4*}[\beta] = 0$. Diagram (3.3) holds.

Now the assertion follows from the commutative diagram

$$\begin{tikzcd}
C^{2n} \arrow[r, "p_3"] \arrow[d] & P^{2n}(2) \arrow[d, "\eta\wedge\mathrm{id}"] & \\
P^{2n-1}(8) \arrow[r, "\rho"] \arrow[d] & P^{2n-1}(2) \arrow[r, "S_2|"] \arrow[d] & \Omega P^{n+1}(2) \\
S^{2n-1} \arrow[r, "[4]"] & S^{2n-1} \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

together with the fact [9] that $[\lambda] \in \pi_{2n}(P^{n+1}(2))$ is of order 4 when $n+1 \equiv 0 \mod 4$.

□

4. Some Lemmas on $P^{2n}(2)$

In this section, we give some lemmas related to the Whitehead products.

Lemma 4.1. *Let j_{2n+1} be the composite*

$$\mathbb{R}\mathrm{P}^{2n} \hookrightarrow SO(2n+1) \longrightarrow \Omega^{2n+1}S^{2n+1}.$$

Then

$$\Omega^{2n+1}([2]) \circ j_{2n+1} \simeq 2 \circ j_{2n+1}.$$

Proof. By [7, Proposition 4.3], the maps $\Omega[2], 2\colon \Omega S^{2n+1} \to \Omega S^{2n+1}$ differ by the homotopy class represented by the composite

$$\Omega S^{2n+1} \xrightarrow{H_2} \Omega S^{4n+1} \xrightarrow{\Omega\omega_{2n+1}} \Omega S^{2n+1}.$$

From the commutative diagram

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{k} \arrow[r, hook] \arrow[d] & \mathbb{R}\mathrm{P}^{k+1} \arrow[d] \\
\Omega^{k+1}S^{k+1} \arrow[r] & \Omega^{k+2}S^{k+2},
\end{tikzcd}$$

there is a commutative diagram

$$\begin{tikzcd}
S^{2k+1} \arrow[r, "\Sigma^{k+1}q_k"] \arrow[d] & \Sigma^{k+1}\mathbb{R}\mathrm{P}^k \arrow[r, hook] \arrow[d, "j_{k+1}"] & \Sigma^{k+1}\mathbb{R}\mathrm{P}^{k+1} \arrow[r, "\Sigma k+1 p_{k+1}"] \arrow[d, "j_{k+2}"] & S^{2k+2} \arrow[d, "\theta_{k+2}"] \\
\Omega^2 S^{2k+3} \arrow[r, "P"] & S^{k+1} \arrow[r, "E"] & \Omega S^{k+2} \arrow[r, "H"] & \Omega S^{2k+3},
\end{tikzcd}$$

where $q_k\colon S^k \to \mathbb{R}\mathrm{P}^k$ is the projection map, $p_{k+1}\colon \mathbb{R}\mathrm{P}^{k+1} \to S^{k+1}$ is the pinch map, the top row is a cofibre sequence, and the bottom row is the EHP sequence. The map

$$\theta_{2k+2}\colon H_{2k+2}(S^{2k+2}) \longrightarrow H_{2k+2}(\Omega S^{2k+3})$$

is an isomorphism [12, Theorem (1.1)]. It follows that there is a commutative diagram

$$\begin{tikzcd}
\Sigma^{2n}\mathbb{R}\mathrm{P}^{2n} \arrow[r, "\Sigma^{2n}p_{2n}"] \arrow[d, "j'_{2n+1}"] & S^{4n} \arrow[r, "\Sigma^{2n}q_{2n}"] \arrow[d, hook, "\theta_{4n}"] & \Sigma^{2n}\mathbb{R}\mathrm{P}^{2n} \arrow[d, "j'_{2n+1}"] \\
\Omega S^{2n+1} \arrow[r, "H"] & \Omega S^{4n+1} \arrow[r, "\Omega\omega_{2n+1}"] & \Omega S^{2n+1}.
\end{tikzcd}$$

We check that the composite

$$\Sigma^2\mathbb{R}\mathrm{P}^{2n} \xrightarrow{\Sigma^2 p_{2n}} S^{2n+2} \xrightarrow{\Sigma^2 q_{2n}} \Sigma^2\mathbb{R}\mathrm{P}^{2n}$$

is null homotopic. If so, the assertion will follow from the above commutative diagram.

Consider the Hopf map

$$H\colon \Sigma\mathbb{R}\mathrm{P}^\infty \wedge \mathbb{R}\mathrm{P}^\infty \longrightarrow \Sigma\mathbb{R}\mathrm{P}^\infty.$$

The composite

$$\Sigma\mathbb{R}\mathrm{P}^{2n} \wedge \mathbb{R}\mathrm{P}^1 \hookrightarrow \Sigma\mathbb{R}\mathrm{P}^\infty \wedge \mathbb{R}\mathrm{P}^\infty \xrightarrow{H} \Sigma\mathbb{R}\mathrm{P}^\infty$$

maps into $\Sigma\mathbb{R}\mathrm{P}^{2n+1}$ by the skeleton reasons. Let

$$f\colon \Sigma^2\mathbb{R}\mathrm{P}^{2n} = \Sigma\mathbb{R}\mathrm{P}^{2n} \wedge \mathbb{R}\mathrm{P}^1 \longrightarrow \Sigma\mathbb{R}\mathrm{P}^{2n+1}$$

be the resulting map. Recall that the mod 2 homology $H_*(\mathbb{R}\mathrm{P}^\infty) = \Gamma(u)$ with $|u| = 1$ is the divided algebra. The Hopf map H induces

$$H_*(\Sigma(\gamma_{2n}(u) \otimes \gamma_1(u))) = \Sigma\gamma_{2n+1}(u).$$

Thus

$$f_*\colon H_{2n+2}(\Sigma^2\mathbb{R}\mathrm{P}^{2n}; \mathbb{Z}/2) \longrightarrow H_{2n+2}(\Sigma\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}/2)$$

is an isomorphism. It follows that the pinch map $\Sigma^2 p_{2n}\colon \Sigma^2\mathbb{R}\mathrm{P}^{2n} \to S^{2n+2}$ lifts to $\Sigma\mathbb{R}\mathrm{P}^{2n+1}$ by f. From the cofibre sequence

$$\Sigma\mathbb{R}\mathrm{P}^{2n+1} \xrightarrow{\Sigma p_{2n+1}} S^{2n+2} \xrightarrow{\Sigma^2 q_{2n}} \Sigma^2\mathbb{R}\mathrm{P}^{2n},$$

we obtain that $\Sigma^2(q_{2n} \circ p_{2n})$ is null homotopic. $\square$

Lemma 4.2. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n} \arrow[rr, "{[2]}"] \arrow[dr, hook] \arrow[dd, "j'_{2n+1}"'] & & \Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n} \arrow[rr] \arrow[dr, hook] \arrow[dd, "j'_{2n+1}"' near start] & & \mathbb{R}\mathrm{P}^{2n} \wedge P^{2n+2}(2) \arrow[dr, hook] \arrow[dd, "k_{2n+2}"' near start] & \\
& \Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n+2} \arrow[rr] \arrow[dd, "j'_{2n+3}" near start] & & \Sigma^{2n+1}\mathbb{R}\mathrm{P}^{2n+2} \arrow[rr] \arrow[dd, "j'_{2n+3}" near start] & & \mathbb{R}\mathrm{P}^{2n+2} \wedge P^{2n+2}(2) \arrow[dd, "k'_{2n+4}"] \\
S^{2n+1} \arrow[rr, "{[2]}" near start] \arrow[dr, "\Sigma^2"] & & S^{2n+1} \arrow[rr] \arrow[dr, "\Sigma^2"] & & P^{2n+2}(2) \arrow[dr] & \\
& \Omega^2 S^{2n+1} \arrow[rr, "\Omega^2({[2]})"] & & \Omega^2 S^{2n+3} \arrow[rr] & & \Omega^2 P^{2n+4}(2)
\end{tikzcd}$$

Proof. By Lemma 4.1, the left cube is homotopy commutative. By using the property of cofibre sequences, there is a map $k_{2n+2} \to \mathbb{R}\mathrm{P}^{2n} \wedge P^{2n+2}(2) \to P^{2n+2}(2)$ so that the right cube in the diagram commutes up to homotopy. $\square$

Mark Mahowald has a result [13, Theorem (1.1.2a)] that $[\iota_{4n-1}, \eta_{4n-1}] = 0$. For mod 2 Moore spaces, we have the following lemma.

Lemma 4.3. *There exists a map $\delta_{4n}\colon P^{8n-2}(2) \to P^{4n}(2)$ with the following properties:*

(1). *There is a homotopy commutative diagram*

$$\begin{tikzcd}
P^{8n-2}(2) \arrow[r, "\delta_{4n}"] & P^{4n}(2) \\
S^{8n-3} \arrow[u, hook] \arrow[r, "\omega_{4n-1}"] & S^{4n-1}. \arrow[u, hook]
\end{tikzcd}$$

(2). *The composite*

$$P^{8n-2}(2) \xrightarrow{\delta_{4n}} P^{4n}(2) \xrightarrow{\Sigma^2} \Omega^2 P^{4n+2}(2)$$

is null homotopic.

(3). *The composite*

$$P^{8n-1}(2) \xrightarrow{\eta\wedge \mathrm{id}} P^{8n-2}(2) \xrightarrow{\delta_{4n}} P^{4n}(2)$$

is null homotopic.

Proof. Let W^n_k be the homotopy of the homotopy fibre of the inclusion map $S^n \to \Omega^k S^{n+k}$, and let $W^n_k(2)$ be the homotopy fibre of the inclusion map $P^n(2) \to \Omega^k P^{n+k}(2)$. By Lemma 3.2, there is a homotopy commutative diagram,

$$\begin{tikzcd}
W^{4n}_\infty(2) \arrow[r] & P^{4n}(2) \arrow[r] & Q(P^{4n}(2)) \\
\mathbb{RP}^\infty_{4n-1}\wedge P^{4n-1}(2) \arrow[u] \arrow[r, "\partial_{4n-2}\wedge\mathrm{id}"] & \mathbb{RP}^{4n-2}\wedge P^{4n}(2) \arrow[u, "k_{4n}"] \arrow[r] & \mathbb{RP}^\infty\wedge P^{4n}(2) \arrow[u] \\
\Sigma^{4n-2}\mathbb{RP}^\infty_{4n-1} \arrow[u, hook] \arrow[d] \arrow[r, "\Sigma^{4n-2}\partial_{4n-2}"] & \Sigma^{4n-1}\mathbb{RP}^{4n-2} \arrow[u, hook] \arrow[d, "j_{4n-1}"] \arrow[r] & \Sigma^{4n-1}\mathbb{RP}^\infty \arrow[u, hook] \arrow[d, "j'_\infty"] \\
W^{4n-1}_\infty \arrow[r] & S^{4n-1} \arrow[r] & Q(S^{4n-1})
\end{tikzcd}$$

with a canonical morphism of fibre sequences from the bottom row to the top row for making a homotopy commutative diagram of cubic diagrams, where $\partial_k\colon \mathbb{RP}^\infty_{k+1} = \mathbb{RP}^\infty/\mathbb{RP}^k \to \Sigma\mathbb{RP}^k$ is the boundary map. Let $\delta_{4n}\colon P^{8n-2}(2) \to P^{4n}(2)$ be the composite

$$S^{4n-1}\wedge P^{4n-1}(2) \hookrightarrow \mathbb{RP}^\infty_{4n-1}\wedge P^{4n-1}(2) \xrightarrow{\partial_{4n-2}\wedge\mathrm{id}} \mathbb{RP}^{4n-2}\wedge P^{4n}(2) \xrightarrow{k_{4n}} P^{4n}(2).$$

From the above commutative diagram, $\delta_{4n}|\colon S^{8n-3} \to P^{4n}(2)$ is homotopic to the composite

$$S^{8n-3} \xrightarrow{\omega_{4n-1}} S^{4n-1} \hookrightarrow P^{4n}(2).$$

Moreover the composite $P^{8n-2}(2) \xrightarrow{\delta_{4n}} P^{4n}(2) \longrightarrow Q(P^{4n}(2))$ is null homotopic by the construction. It follows that $\Sigma^2\delta_{4n}\colon P^{8n}(2) \to P^{4n+2}(2)$ is null homotopic by dimensional reasons.

Now we check condition (3) in the statement. Observe that the reduced mod 2 homology of $\mathbb{RP}^\infty_{4n-1}$ has a basis $\{u^k\}$ with $k \geq 4n-1$. The Steenrod operation

$$Sq^2(u^{4k-1}) = Sq(u^{4k-4} \cdot u^3) = u^{4k-4}Sq^2(u^3) = u^{4k+1}.$$

Thus

$$Sq^2_*\colon H_{8n-1}(\Sigma^{4n-2}\mathbb{RP}^\infty_{4n-1}) = \mathbb{Z}/2 \longrightarrow H_{8n-3}(\Sigma^{4n-2}\mathbb{RP}^\infty_{4n-1}) = \mathbb{Z}/2$$

is an isomorphism. It follows that the composite

$$S^{8n-2} \xrightarrow{\eta} S^{8n-3} \hookrightarrow \Sigma^{4n-2}\mathbb{RP}^\infty_{4n-1}$$

is null homotopic. By smashing with mod 2 Moore spaces, the composite

$$P^{8n-1}(2) \xrightarrow{\eta\wedge\mathrm{id}} P^{8n-2}(2) \hookrightarrow \Sigma^{4n-2}\mathbb{RP}^\infty_{4n-1} \wedge P^{4n-1}(2)$$

is null homotopic. Condition (3) is satisfied and hence the result.

□

5. Proof of Theorem 1.1

We use the notation $W^n_k(2)$ defined in the proof of Lemma 4.3. Consider the homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
\Omega W^{2n}_\infty \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & W^{2n}_\infty(2) \arrow[r] \arrow[d] & W^{2n}_\infty \arrow[d] \\
\Omega S^{2n} \arrow[r, "\partial"] \arrow[d] & F^{2n}\{2\} \arrow[r] \arrow[d] & P^{2n}(2) \arrow[r, "\mathrm{pinch}"] \arrow[d] & S^{2n} \arrow[d] \\
\Omega Q(S^{2n}) \arrow[r, "2"] & Q(S^{2n-1}) \arrow[r] & Q(P^{2n}(2)) \arrow[r] & Q(S^{2n}).
\end{tikzcd}$$

For a space X, let $\{P^n(2), X\} = [P^n(2), Q(X)]$ denote the group of stable homotopy classes from $P^n(2)$ to X.

Lemma 5.1. (1). *The stabilization* $[P^{4n-2}(2), S^{2n}] \to \{P^{4n-2}(2), S^{2n}\}$ *is an isomorphism.*

(2). *The stablization* $[P^{4n-2}(2), \Omega S^{2n}] \to [P^{4n-2}(2), \Omega Q(S^{2n})]$ *is onto.*

(3). *Let $4n \neq 4, 8$. Then the kernel of $[P^{8n-2}(2), S^{4n-1}] \to \{P^{8n-2}(2), S^{4n-1}\}$ is $Z/2$ generated by any map $\phi\colon P^{8n-2}(2) \to S^{4n-1}$ such that $\phi|_{S^{8n-3}}$ is the Whitehead square.*

Proof. Assertions (1) and (2) follows immediately from the fact that S^{2n} is the $(4n-1)$-skeleton of ΩS^{2n+1}. For assertion (3), consider homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
\Omega^2 S^{8n-1} \arrow[r, equal] \arrow[d] & \Omega^2 S^{8n-1} \arrow[d, "P"] & \\
W_2^{4n-1} \arrow[r] \arrow[d] & S^{4n-1} \arrow[r, "\Sigma^2"] \arrow[d, "E"] & \Omega^2 S^{4n+1} \arrow[d, equal] \\
\Omega^3 S^{8n+1} \arrow[r, "\Omega P"] \arrow[d] & \Omega S^{4n} \arrow[r, "\Omega E"] \arrow[d, "H"] & \Omega^2 S^{4n+1} \\
\Omega S^{8n-1} \arrow[r, equal] & \Omega S^{8n-1}. &
\end{tikzcd}$$

Since the composite

$$S^{8n-2} \hookrightarrow \Omega^3 S^{8n+1} \xrightarrow{\Omega P} \Omega S^{4n} \xrightarrow{H} \Omega S^{8n-1}$$

is of degree 2, we have

$$\mathrm{sk}_{8n-1}(W_2^{4n-1}) = P^{8n-2}(2).$$

It follows that there is an exact sequence

$$[P^{8n-2}(2), \Omega^3 S^{4n+1}] \to [P^{8n-2}(2), P^{8n-2}(2)] \to [P^{8n-2}(2), S^{4n-1}] \to \{P^{8n-2}(2), S^{4n-1}\}.$$

By the proof of Lemma 4.3, $\omega_{4n-1} \circ \eta$ is null homotopic. Thus the composite

$$P^{8n-2}(2) \longrightarrow S^{8n-2} \xrightarrow{\eta} S^{8n-3} \longrightarrow P^{8n-2}(2) \longrightarrow S^{4n-1}$$

is null homotopic, and so the image of $[P^{8n-2}(2), P^{8n-2}(2)] = \mathbb{Z}/4$ in $[P^{8n-4}(2), S^{4n-1}]$ is $\mathbb{Z}/2$. The proof is finished. $\square$

Lemma 5.2. *There is a homotopy decomposition*

$$\Omega F^{2n}\{2\} \simeq \Omega S^{2n-1} \times \Omega S^{4n-2} \times \Omega P^{6n-3}(2)$$

up to dimension $8n-8$.

Proof. Consider the homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
\Omega S^{2n} \arrow[r] \arrow[d, equal] & S^{2n-1} \arrow[r] & \tau(S^{2n}) \arrow[r] & S^{2n} \arrow[d, equal] \\
\Omega S^{2n} \arrow[r] & F^{2n}\{2\} \arrow[r] \arrow[u] & P^{2n}(2) \arrow[r] \arrow[u, hook] & S^{2n} \\
& Y \arrow[r, equal] \arrow[u] & Y, \arrow[u] &
\end{tikzcd}$$

where $\tau(S^{2n}) = SO(2n+1)/SO(2n-1) = V_{2,2n+1}$ is the 2-frame Stiefel manifold. Since the map $F^{2n}\{2\} \to S^{2n-1}$ admits a cross-section, there is a homotopy decomposition

$$\Omega F^{2n}\{2\} \simeq \Omega Y \times \Omega S^{2n-1}.$$

Let $f\colon \Omega S^{4n-2} \to \Omega Y$ be the extension of the inclusion of the bottom cell. Let $g\colon \Omega P^{6n-3}(2) = \Omega\Sigma L_3(P^{2n-1}(2)) \to \Omega Y$ be the map in the functorial decomposition of

$$\Omega\Sigma X \simeq \Omega\Sigma L_3(X) \times ?$$

for 2-local spaces. Then the map

$$\Omega S^{4n-2} \times \Omega P^{6n-3}(2) \xrightarrow{(f,g)} \Omega Y$$

induces an isomorphism on homology up to dimension $8n-8$. The proof is finished.

$\square$

We restate Theorem 1.1 as follows.

Theorem 5.3. *Let $n > 1$ The power map $4\colon \Omega^2 P^{4n}(2) \to \Omega^2 P^{4n}(2)$ restricted to the skeleton $\mathrm{sk}_{4(4n-2)-1}(\Omega^2 P^{4n}(2))$ is null homotopic.*

Proof. If the Whitehead square ω_{4n-1} is divisible by 2, we have proved the assertion in Theorem 3.6. Now we assume that ω_{4n-1} is not divisible by 2. Similar to the situation in the proof of Theorem 3.6, it suffices to prove that the composite

(5.1)

$$\Sigma D_2 \xrightarrow{\bar{\sigma}_2} (P^{4n-1}(2)^{\wedge 2} \xrightarrow{\text{pinch}} P^{8n-2}(2) \xrightarrow{\eta\wedge\mathrm{id}} P^{8n-3}(2) \xrightarrow{S_2|} \Omega P^{4n}(2)$$

is null homotopic. Our proof is given by controlling the map

$$S_2|\colon P^{8n-1}(2) \to \Omega P^{4n}(2).$$

By Lemma 5.2, $\Omega F^{4n}\{2\}\simeq \Omega S^{4n-1}\times\Omega S^{8n-2}$ up to dimension $15n-5$. Thus

$$[P^{8n-2}(2),F^{4n}\{2\}]\cong[P^{8n-2}(2),S^{4n-1}]\oplus[P^{8n-2}(2),S^{8n-2}]=[P^{8n-2}(2),S^{4n-1}]\oplus\mathbb{Z}/2.$$

By (3) of Lemma 3.4, there is an exact sequence

$$\mathbb{Z}/2\longrightarrow[P^{8n-2}(2),S^{4n-1}]\longrightarrow\{P^{8n-2}(2),S^{4n-1}\}.$$

By assertions (1) and (2) of Lemma 5.1, $\mathrm{Ker}([P^{8n-2}(2),P^{4n}(2)]\to\{P^{8n-2}(2),P^{4n}(2)\})$ is contained in

$$\mathrm{Im}(\mathbb{Z}/2\oplus\mathbb{Z}/2\to[P^{8n-2}(2),F^{4n}\{2\}]\to[P^{8n-2}(2),P^{4n}(2)].$$

Let $\bar\lambda_{4n}\colon P^{8n-2}(2)\to P^{4n}(2)$ be the composite

$$P^{8n-2}(2)\xrightarrow{pinch}S^{8n-2}\xrightarrow{\lambda_{4n}}F^{4n}\{2\}\longrightarrow P^{4n}(2),\tag{5.2}$$

where $\lambda_{4n}\colon S^{8n-2}\to F^{4n}\{2\}$ inducing an isomorphism on H_{8n-2}. Since $\bar\lambda_{4n}|_{S^{8n-3}}$ is trivial but $\delta_{4n}|_{8n-3}$ is essential (because ω_{4n-1} is not divisible by 2), $[\bar\lambda_{4n}]\neq[\delta_{4n}]$. Thus the elements $\{[\bar\lambda_{4n}],[\delta_{4n}]\}$ generates a subgroup of $[P^{8n-2}(2),P^{4n}(2)]$ containing $\mathrm{Ker}([P^{8n-2}(2),P^{4n}(2)]\to\{P^{8n-2}(2),P^{4n}(2)\})$. Since

$$[S_2|]\in\mathrm{Ker}([P^{8n-2}(2),P^{4n}(2)]\to\{P^{8n-2}(2),P^{4n}(2)\}),$$

we have $[S_2|]=[\bar\lambda_{4n}]$, $[\delta_{4n}]$ or $[\bar\lambda_{4n}+\delta_{4n}]$.

By Lemma 4.3,

$$\delta_{4n}\circ(\eta\wedge\mathrm{id})\colon P^{8n-1}(2)\longrightarrow P^{4n}(2)$$

is null homotopic.

Following the lines in the proof of Theorem 3.6, with using the properties that $\bar\lambda_{4n}$ factors through S^{8n-3} by (5.2) and any map $S^{8n-3}\to\Omega P^{4n}(2)$ having nontrivial Hurewicz image is of order 4 [9], the composite (5.1) is null homotopic if $S_2|$ is replaced by $\bar\lambda'_{4n}\colon P^{8n-3}(2)\to\Omega P^{4n}(2)$. The proof is finished. $\square$

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